Eigenvectors and Eigenvalues

Defn: Let A be an $n \times n$ matrix and X a non-zero vector of $\mathbb{R}^n$. We say X is an eigenvector of A if there exists a scalar λ such that

$$Ax = \lambda \cdot X$$

If X is an eigenvector, then the scalar λ is called an eigenvalue of A. In this case we say x is the eigenvector associated to eigenvalue λ.

Example

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and $u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

- $Au = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

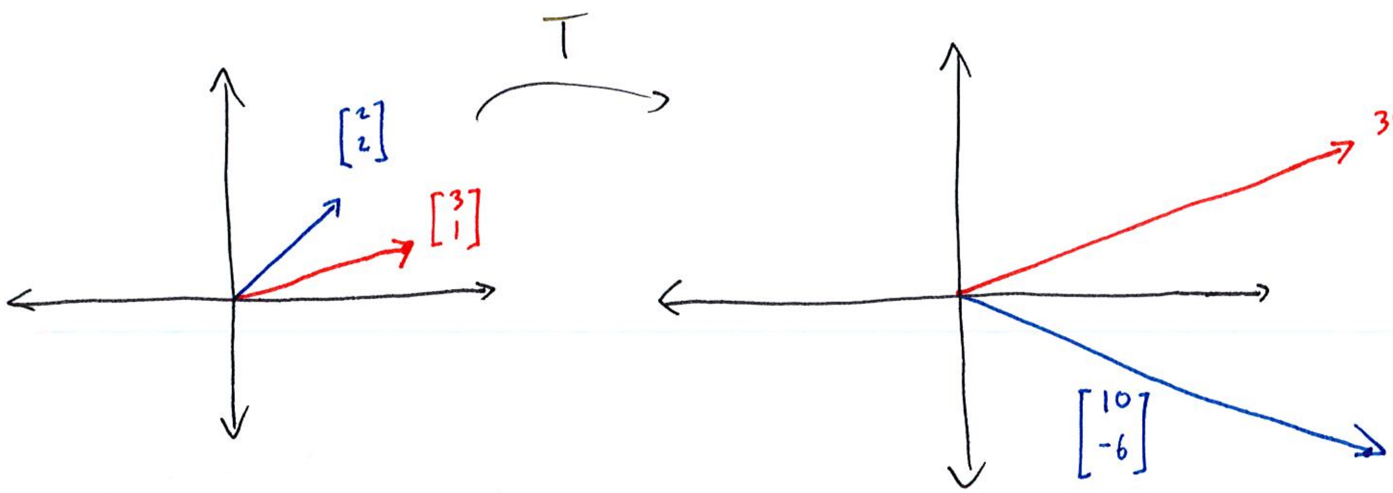
Thus $u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of A and it corresponds to eigenvalue $\lambda = 3$.

- $Av = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$ Not a multiple of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Thus v is not an eigenvector of A.

Thinking of A as a linear transformation ($Tx = Ax$), the eigenvectors are exactly the vectors which don't change direction under T (might go backwards but that's ok).

For instance, take $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $Tx = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} x$



$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$ go same direction!

Question: Given a matrix, how do we find eigenvectors and eigenvalues?

Usually we find eigenvalues first and then find corresponding eigenvectors.

- More on this in §5.2

Example

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ as before. Show -7 is an eigenvalue of A and find a corresponding eigenvector.

Solution: We must show that there is some vector X with $Ax = -7x$. Equivalently, we show there's an nonzero x with

$$(A + 7I_2)x = 0$$

In other words show $A + 7I$ has a nontrivial null space and find a basis for it!

$$A + 7I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = -\frac{1}{3}x_2 \\ x_2 \text{ free} \end{cases}$$

i.e. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue -7. Notice $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is too! Makes sense since $\begin{bmatrix} 1 \\ -3 \end{bmatrix} = -3\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$, hence

$$\mathrm{Nul}(A + 7I) = \mathrm{span}\left\{\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$$

In general, if A is an $n \times n$ matrix then λ is an eigenvalue if and only if

$$(A - \lambda I)x = 0$$

has a nontrivial solution. If λ is an eigenvalue, then the null space of $A - \lambda I$ is called the eigenspace of A associated to λ.

This is a subspace of $\mathbb{R}^n$ and is spanned by the eigenvectors corresponding to λ.

Remark

With the above, we see that in order to find eigenvalues λ, we must find the values so that

$$(A - \lambda I)x = 0$$

has a nontrivial solution. Since $A - \lambda I$ is a square matrix, this is the same as $A - \lambda I$ not being invertible.

What's a quick way to check this for square matrices?

Think about it!

Theorem

Let A be an upper (or lower, doesn't matter) triangular ~~mat~~ square matrix. The eigenvalues of A are exactly the entries on the diagonal.

Proof

We need $(A-\lambda I)x = 0$ to have a nontrivial solution. In other words when is $A-\lambda I$ not invertible?

Use your answer to written Hw #11 §2.2/2.3 pg. 124 #21.

Notice $A-\lambda I$ is also triangular!

Example

The eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 7 & 1 & 2 \end{bmatrix}$

are 0 and 2. Moreover these are the only eigenvalues!

Remark

$\lambda = 0$ is an eigenvalue if and only if $Ax = 0x = 0$ for nonzero x. In other words $\lambda = 0$ is an eigenvalue if and only if A is not invertible!

Theorem

Let A be an $n \times n$ matrix and suppose $v_1, v_2, \ldots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. The set $\{v_1, \ldots, v_r\}$ is linearly independent.